Увод в теорията на операторни алгебри с приложение в математиката и физиката
Introduction to Theory of Operator Algebras with Applications to Mathematics and Physics
Nikolay M. Nikolov
Lecture 2

Elementary theory of C^*-algebras (C^*-алгебра)

Def. (a) Involutive algebra (инволютивна алгебра), or also $*$-algebra, or star-algebra (звезда-алгебра) $\mathcal{A}$ is a complex associative algebra endowed with (снабдена с) a map $a \mapsto a^*$ ($\forall a \in \mathcal{A}$) which is:

antilinear: $(\alpha a + \beta b)^* = \bar{\alpha} a^* + \bar{\beta} b^*$ $(\forall a, b \in \mathcal{A})$

involutive (or, involution): $(a^*)^* = a$ $(\forall a \in \mathcal{A})$

antimorphism: $(a \cdot b)^* = b^* \cdot a^*$ $(\forall a, b \in \mathcal{A})$

Morphism of $*$-algebras is a morphism $\varphi : \mathcal{A} \to \mathcal{B}$ s.t.

$$\varphi(a^*) = \varphi(a)^* \quad (\forall a \in \mathcal{A})$$

<u>Note</u>: $1^* = 1$ since $1^* a = 1^* a^{**} = (a^* 1)^* = a^{**} = a$ and

$$a\, 1^* = a^{**} 1^* = (1\, a^*)^* = a^{**} = a$$

(b) C^*-algebra (C^*-алгебра) is a Banach algebra s.t.

$$\|a^* \cdot a\| = \|a\|^2 \quad (\forall a \in \mathcal{A}) \quad \text{- the } C^*\text{-norm identity}$$

<u>Note</u>: As a consequence we have: $\|a\| = \|a^*\|$ $(\forall a \in \mathcal{A})$:

$$\|a\|^2 = \|a^* \cdot a\| \leqslant \|a^*\| \, \|a\| \Rightarrow \|a\| \leqslant \|a^*\| \quad (\forall a \in \mathcal{A}).$$

Applying this for a^* we get $\|a^*\| \leqslant \|a^{**}\| = \|a\|$

Another consequence is: $\|1\| = 1$ since $\|1\| = \|1^* . 1\| = \|1\|^2$ and $\|1\| \neq 0$ if $\mathcal{A} \neq 0$ (and then $1 \neq 0$).

(c) Involutive normed/Banach algebra is an involutive algebra, which is at the same time a normed/Banach algebra, s.t.:

$$\|a\| = \|a^*\|$$

Example. We know from Lecture 1 that Bounded Maps (M, V) is a Banach space for every Banach space V. If $V = \mathbb{C}$ this is a Banach algebra with the point-wise (поточково) multiplication:

$$(F \cdot G)(p) := F(p)\, G(p) \qquad (\forall p \in M)$$

Let us check the Banach inequality:

$$\|F \cdot G\| = \sup_{p \in M} |F(p)\, G(p)| \leq \left(\sup_{p \in M} |F(p)| \right) \left(\sup_{q \in M} |G(q)| \right) = \|F\| \, \|G\|$$

If we set (ако положим) $F^*(p) := \overline{F(p)}$ for every $p \in M$ then the Banach algebra Bounded Maps $(M, \mathbb{C})$ becomes a C^*-algebra:

$$\|F \cdot F^*\| = \sup_{p \in M} |F(p) \cdot \overline{F(p)}| = \sup_{p \in M} |F(p)|^2 = \left(\sup_{p \in M} |F(p)| \right)^2 = \|F\|^2$$

Exercise Suggest other examples when the Banach space Bounded Maps (M, V) has a structure of a C^*-algebra.

Another example of a C^*-algebra is the algebra of all bounded linear operators in a Hilbert space. But we shall prove this later in this lecture course.

Functional calculus in a Banach algebra and spectra of elements
Функционално смятане в Банахова алгебра и спектри на елементи

Lemma 2.1. The series $\sum_{n=1}^{\infty} v_n$ in a Banach space V is convergent, i.e. $\exists \lim_{N \to \infty} \sum_{n=1}^{N} v_n$, if (a sufficient condition) the series $\sum_{n=1}^{\infty} \|v_n\|$ converges.

Hint for the proof. $\left\| \sum_{n=j}^{k} v_n \right\| \leq \sum_{n=j}^{k} \|v_n\|$

Def. Series in a Banach space satisfying the condition of Lemma 2.1. are called absolutely convergent (абсолютно сходящи редове).

Corollary 2.2. For an absolutely convergent series $\sum_{n=1}^{\infty} v_n$ for every change of enumeration $\sigma : \mathbb{N} \cong \mathbb{N}$ the series $\sum_{n=1}^{\infty} v_{\sigma(n)}$ is also absolutely covergent and $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} v_{\sigma(n)}$.

Corollary 2.3 (a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two absolutely convergent series in a Banach algebra $\mathcal{A}$. Then the series $\sum_{m,n=1}^{\infty} a_m \cdot b_n$ is also absolutely convergent with respect to any way of enumerating it (по отношение на всеки начин на номериране) and

$$\sum_{m,n=1}^{\infty} a_m \cdot b_n = \left(\sum_{n=1}^{\infty} a_n\right) \cdot \left(\sum_{n=1}^{\infty} b_n\right)$$

(b) Let $\sum_{n=0}^{\infty} \alpha_n z^n$ be a complex power series that is absolutely convergent for $|z| < r$. Then if $\mathcal{A}$ is a unital Banach algebra and $\|a\| < r$ then the series

$$\sum_{n=0}^{\infty} \alpha_n a^n := \alpha_0 1_{\mathcal{A}} + \alpha_1 a + \alpha_2 a^2 + \cdots$$ is absolutely convergent.

Proof. (a) $\sum_{m,n=1} \|a_m \cdot b_n\| \leq \sum_{m,n=1} \|a_m\| \|b_n\| = \left(\sum_{n=1}^{\infty} \|a_n\|\right) \cdot \left(\sum_{n=1}^{\infty} \|b_n\|\right)$

(b) The series $\sum_{n=0}^{\infty} |\alpha_n| \|a\|^n$ is convergent by assumption; on the other hand, $\|a^n\| \leq \|a\|^n$ by induction from the Banach inequality. □

Corollary 2.4. Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$ with $\|a\| < 1$. Then the series

$$b = \sum_{n=0}^{\infty} a^n \quad \text{and} \quad c = \sum_{n=0}^{\infty} \binom{1/2}{n} a^n$$ absolutely converge

and $b \cdot (1-a) = (1-a) \cdot b = 1$, $c^2 = 1 + a$,

i.e. $b = (1-a)^{-1}$ and $c = (1+a)^{1/2}$.

<u>Proof.</u> For $\forall \gamma \in \mathbb{C}$ the complex power series $\sum_{n=0}^{\infty} \binom{\gamma}{n} z^n = (1+z)^\gamma$ is absolutely convergent for $|z| < 1$. So, apply Corollary 2.3(b).

The identities $(1-a) \cdot \left(\sum_{n=0}^{\infty} a^n \right) = 1$ and $\left(\sum_{n=0}^{\infty} \binom{1/2}{n} a^n \right)^2 = 1 + a$

follow first from purely algebraic identities:

$$\left(\sum_{n=0}^{\infty} \binom{\gamma_1}{n} z^n \right) \left(\sum_{n=0}^{\infty} \binom{\gamma_2}{n} z^n \right) = \sum_{n=0}^{\infty} \binom{\gamma_1+\gamma_2}{n} z^n$$

– these are identities of formal power series and they are generating series of algebraic identities between the binomial coefficients (they follow from $(1+z)^{\gamma_1} (1+z)^{\gamma_2} = (1+z)^{\gamma_1+\gamma_2}$, for $|z|<1$); second, the series converge absolutely due to Corollary 2.3(a). $\square$

<u>Def.</u> Spectrum of an element (спектър на елемент) $a \in \mathcal{A}$ of a unital Banach algebra $\mathcal{A}$

$$\sigma_{\mathcal{A}}(a) := \{ z \in \mathbb{C} \mid \nexists\, c \in \mathcal{A} \text{ s.t. } c \cdot (a - z.1_{\mathcal{A}}) = (a - z.1_{\mathcal{A}}) \cdot c = 1_{\mathcal{A}} \}.$$

If the algebra $\mathcal{A}$ does not have a unit then we still can define a spectrum of an element by "adding a unit" to the algebra, which is a construction that we shall consider later.

Resolvent (резолвента): $R_a(z) = (a - z.1)^{-1}$, $\forall z \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$

resolvent identity (резолвентно тъждество):

$$R_a(z) - R_a(w) = (z-w) R_a(z) R_a(w) \quad \forall z, w \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$$

$$\left(\Leftarrow \ (a - z.1)^{-1} \,\Big|\, (a - w.1 - a + z.1) = (z-w).1 \,\Big|\, (a - w.1)^{-1} \right)$$

In particular, $R_a(z)$ is a continuos function in z (and even, analytic).

<u>Note:</u> $\|a^{-1} - b^{-1}\| \leq \|a^{-1}\| \, \|a-b\| \, \|b^{-1}\| \Longrightarrow$ $a \mapsto a^{-1}$ is a continuous map

$$\|a^{-1}\| \leq \|a^{-1} - b^{-1}\| + \|b^{-1}\| \Longrightarrow \|a^{-1}\| \leq \frac{\|b^{-1}\|}{1 - \|b^{-1}\| \, \|a-b\|}$$

<u>Theorem 2.5.</u> In every (unital) Banach algebra $\mathcal{A}$ the spectrum $\sigma_{\mathcal{A}}(a)$ is a nonempty compact subset of $\mathbb{C}$ for every $a \in \mathcal{A}$. Furthermore,

$$\rho_{\mathcal{A}}(a) := \sup\{|z| \mid z \in \sigma_{\mathcal{A}}(a)\} \leq \|a\|.$$

$\rho_{\mathcal{A}}(a)$ is called spectral radius of a (спектрален радиус на a).

<u>Proof.</u> compact set in $\mathbb{C}$ = closed and bounded set.

i) $\sigma_{\mathcal{A}}(a)$ is closed, or equivalently, $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ is open.

$\mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ is the domain where the resolvent exists. Then

$$R_a(w) = \Big(a - z.1 + (z-w).\underbrace{R_a(z)\cdot(a-z.1)}_{1}\Big)^{-1} =$$

$$= \Big[(a-z.1)\big(1+(z-w)R_a(z)\big)\Big]^{-1} = \big(1+(z-w)R_a(z)\big)^{-1}R_a(z).$$

$\Rightarrow$ If $z \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ and $\underbrace{\|(z-w)R_a(z)\| < 1}_{\text{i.e. } |z-w| < \|R_a(z)\|^{-1}}$ then $\exists R_a(w)$

i.e., $w \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$. $\Rightarrow$ $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ is open.

<u>Remark</u> Let us point out the exansion we have derived above:

$$R_a(w) = \big(1+(z-w)R_a(z)\big)^{-1}R_a(z) = \sum_{n=0}^{\infty} R_a(z)^{n+1}(z-w)^n$$

which is absolutely convergent for $|z-w| < \|R_a(z)\|^{-1}$.

ii) $\sigma_{\mathcal{A}}(a) \subseteq \{z \in \mathbb{C} \mid |z| \leq \|a\|\}$? $\Leftrightarrow$ $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a) \supseteq \{z \in \mathbb{C} \mid |z| > \|a\|\}$?

$\exists R_a(z) = -z^{-1}\left(1 - \frac{1}{z}a\right)^{-1} \xLeftarrow{\text{Corollary 2.4}} \|\frac{1}{z}a\| < 1$, i.e. if $|z| > \|a\|$.

iii) Why $\sigma_{\mathcal{A}}(a)$ is nonempty?

Let $\varphi : \mathcal{A} \to \mathbb{C}$ be any bounded linear functional on $\mathcal{A}$ i.e. $\varphi \in \mathcal{A}^*$. By the resolvent identity we obtain

$$\varphi(R_a(z)) - \varphi(R_a(w)) = (z-w)\,\varphi(R_a(z)R_a(w))$$

and hence, $\varphi(R_a(z))$ is an analytic function on $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$.

Now if $\sigma_A(a) = \emptyset$ then $\mathbb{C} \setminus \sigma_A(a) = \mathbb{C}$ and $\varphi(R_a(z))$ is an entire function (цяла функция = функция аналитична в $\mathbb{C}$) for every $\varphi \in A^*$. But $\lim\limits_{z \to \infty} \varphi(R_a(z)) = 0$ since:

$$R_a(z) = z^{-1}\left(\frac{1}{z} a - 1\right)^{-1} \xrightarrow[z \to \infty]{} 0 \quad \text{and } \varphi \text{ is continuos.}$$

Hence, by the Liouville's theorem (теоремата на Лиувил) $\varphi(R_a(z)) = 0$ for $\forall \varphi \in A^*$. By the Hahn-Banach (Хан-Банах) theorem[1] this implies $R_a(z) = 0$ for $\forall z \in \mathbb{C}$, which is impossible, since $(a - z.1_A) \cdot R_a(z) = 1$. The latter contradiction implies that $\sigma_A(a) \neq \emptyset$. $\square$

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[1] <u>Hahn-Banach theorem</u> (теорема на Хан-Банах)

Every bounded linear functional $\psi : W \to \mathbb{C}$ defined on a closed linear subspace $W \subseteq V$ of a Banach space V admits a continuation $\varphi : V \to \mathbb{C}$ $(\varphi|_W = \psi)$ to a bounded linear funtional on V such that $\|\varphi\| = \|\psi\|$. $\square$

Here we only use the following corollary:

For every nonzero $v \in V$ in a Banach space V there exists a bounded linear functional $\varphi : V \to \mathbb{C}$ such that $\varphi(v) = 1$.

<u>Def.</u> A division algebra (алгебра с деление) is a unital associative algebra in which every nonzero element is invertible.

<u>Corollary 2.6.</u> If a Banach algebra A is a division algebra then $A = \mathbb{C}$.

<u>Proof.</u> Let $a \in A$ then $\sigma_A(a) \neq \emptyset$. If $z \in \sigma_A(a)$ then $a - z.1_A$ is non-invertible. $\Rightarrow$ $a - z.1_A = 0$, i.e. $a = z.1_A$. $\square$

<u>Remark</u> Over the reals $\mathbb{R}$ there are three division algebras, which are Banach spaces with a continuous multiplication: $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ (quaternions = кватерниони).

Theorem 2.7 Let A be a unital Banach algebra, $a \in A$. Then

$$\rho_A(a) = \inf_{n \geq 1} \|a^n\|^{1/n} = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Lemma 2.8.
(Cauchy-Hadamard theorem / теорема на Коши-Адамар)
Let V be a Banach space and $\sum_{n=0}^{\infty} c_n z^n$, $c_n \in V$, $z \in \mathbb{C}$ be a power series.

(a) If $\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent, i.e. $\sum_{n=0}^{\infty} \|c_n\| \, |z|^n < \infty$, then $|z| \leq \liminf_{n \to \infty} \|c_n\|^{-1/n}$

(b) If $|z| < \liminf_{n \to \infty} \|c_n\|^{-1/n}$ then $\sum_{n=0}^{\infty} c_n z^n$ is absolutely convergent.

Example. $f(z) = \sqrt{1+z} = \sum_{n=0}^{\infty} \binom{1/2}{n} z^n$ is absolutely convergent for $|z| \leq 1$ but for $z = -1$ it has a ramification point.
(точка на разклонение)

Remark The number $\liminf_{n \to \infty} \|c_n\|^{-1/n}$ is called radius of convergence
(радиус на сходимост).

Proof of Lemma 2.8. (a) If $\sum_{n=0}^{\infty} \|c_n\| \, |z|^n < \infty$ then $\exists K > 0$ s.t. $\|c_n\| \, |z|^n < K$ $(\forall n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\})$
(i.e. elements of the series are bouded in norm).

$\Rightarrow \|c_n\|^{1/n} |z| < K^{1/n}$ $(\forall n)$. $\Rightarrow \limsup_{n \to \infty} \|c_n\|^{1/n} |z| \leq 1$.

$\Rightarrow |z| \leq \liminf_{n \to \infty} \|c_n\|^{-1/n}$ (since $x \mapsto x^{-1}$ is decreasing function / намаляваща функция)

(b) If $|z| < \liminf_{n\to\infty} \|c_n\|^{-1/n}$, or equivalenty,
$\limsup_{n\to\infty} \|c_n\|^{1/n}|z| < 1$ then only for finite number of values of $n \in \mathbb{N}$ $\|c_n\|^{1/n}|z| \geq 1$ and for the remaining $\|c_n\|^{1/n}|z| < \lambda < 1$.

$$\Rightarrow \sum_{n=0}^{\infty} \|c_n\| \, |z|^n < \text{Const} + \sum_{n=0}^{\infty} \lambda^n < \infty . \qquad \square$$

<u>Corollary 2.9.</u> If $\mathcal{A}$ is a unital Banach algebra, $a \in \mathcal{A}$ then $\rho_{\mathcal{A}}(a) = \limsup_{n\to\infty} \|a^n\|^{1/n}$

<u>Proof.</u> $\lambda > \limsup_{n\to\infty} \|a^n\|^{1/n} \iff \lambda^{-1} < \liminf_{n\to\infty} \|a^n\|^{-1/n}$

Then the series $\sum_{n=0}^{\infty} a^n z^{-n} = (1 - az^{-1})^{-1} = -z R_a(z)$ is absolutely convergent for $|z|^{-1} \leq \lambda^{-1}$. $\Rightarrow z \notin \sigma_{\mathcal{A}}(a)$ for $\forall |z| \geq \lambda$. $\Rightarrow$ $\lambda \geq \rho(a)$. $\Rightarrow \rho_{\mathcal{A}}(a) \leq \limsup_{n\to\infty} \|a^n\|^{1/n}$.

Conversely, let $\lambda > \rho_{\mathcal{A}}(a)$. $\overset{?}{\Rightarrow}$ $\lambda \geq \limsup_{n\to\infty} \|a^n\|^{1/n}$

For all $|z| \geq \lambda$: $z \notin \sigma_{\mathcal{A}}(a)$, i.e. $\exists R_a(z) = -z^{-1}(1 - az^{-1})^{-1}$.
As in the proof of Theorem 2.5 (part (iii)) it follows that $\varphi((1-aw)^{-1})$ is an analytic function for all $|w| \leq \lambda^{-1}$ and every bounded linear functional $\varphi: \mathcal{A} \to \mathbb{C}$. The Taylor series of $\varphi((1-aw)^{-1})$ is $\sum_{n=0}^{\infty} \varphi(a^n) w^n$ and hence, it is absolutely convergent for $|w| \leq \liminf_{n\to\infty} |\varphi(a^n)|^{-1/n}$.
Thus, $\lambda^{-1} \leq \liminf_{n\to\infty} |\varphi(a^n)|^{-1/n}$, i.e. $1 \geq \limsup_{n\to\infty} \left\| \varphi\left(\frac{a^n}{\lambda^n}\right) \right\|^{-1/n}$ for all bounded $\varphi: \mathcal{A} \to \mathbb{C}$.

In particular, the series $\left\{\varphi\left(\frac{a^n}{\lambda^n}\right)\right\}_{n=0}^{\infty}$ is bounded for every bounded φ.

By the Banach-Steinhaus (Банах-Щайнхаус) theorem*
it follows that:

$\exists K>0$ s.t. $\left\|\frac{a^n}{\lambda^n}\right\| < K \ (\forall n)$. $\Rightarrow$ $\limsup_{n\to\infty} \left\|\frac{a^n}{\lambda^n}\right\|^{1/n} \leq 1$

i.e. $\lambda \geq \limsup_{n\to\infty} \|a^n\|^{1/n}$. $\square$

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*) <u>Banach-Steinhaus (Банах-Щайнхаус) theorem</u>

Let V be a Banach space and $\{c_n\}_{n=1}^{\infty} \subseteq V$. Then if for every bounded linear functional $\varphi: V \to \mathbb{C}$ $\exists \sup_{n\geq 1} |\varphi(c_n)| \ (<\infty)$ then $\exists \sup_{n\geq 1} \|c_n\| \ (<\infty)$.

<u>Lemma 2.10.</u> If $\mathcal{A}$ is a unital Banach algebra, $a\in\mathcal{A}$ then $\limsup_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\geq 1} \|a^n\|^{1/n}$

<u>Proof.</u> It suffices to prove that $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq \inf_{n\geq 1} \|a^n\|^{1/n}$

since $\limsup_{n\to\infty} \|a^n\|^{1/n} \geq \liminf_{n\to\infty} \|a^n\|^{1/n} \geq \inf_{n\geq 1} \|a^n\|^{1/n}$

Let $\lambda > \inf_{n\geq 1} \|a^n\|^{1/n}$. $\overset{?}{\Longrightarrow}$ $\lambda \geq \limsup_{n\to\infty} \|a^n\|^{1/n}$

$\Rightarrow$ $\exists m\geq 1$ s.t. $\lambda \geq \|a^m\|^{1/m}$. Then for $\forall n=1,2,\dots$ $\exists q=0,1,2,\dots$ and $\exists r=0,\dots,m-1$ s.t. $n=qm+r$.

$$\left\|\left(\frac{a}{\lambda}\right)^n\right\| = \left\|\left(\left(\frac{a}{\lambda}\right)^m\right)^q \left(\frac{a}{\lambda}\right)^r\right\| \leq \underbrace{\left\|\left(\frac{a}{\lambda}\right)^m\right\|^q}_{\leq 1} \underbrace{\left\|\left(\frac{a}{\lambda}\right)^r\right\|}_{\leq K \text{ for } r=0,\dots,m-1} \leq K$$

$\Rightarrow \left\|\left(\frac{a}{\lambda}\right)^n\right\|^{1/n} \leq K^{1/n} \ (\forall n)$. $\Rightarrow \limsup_{n\to\infty} \left\|\left(\frac{a}{\lambda}\right)^n\right\|^{1/n} \leq 1$. $\square$

Now Theorem 2.7 follows from Corollary 2.9 and Lemma 2.10.

<u>Corollaries 2.11.</u> (Implications for C^*-algebras.)

Let $\mathcal{A}$ be a unital C^*-algebra, $a \in \mathcal{A}$.

(a) If $a = a^*$ ($\overset{def}{\Longleftrightarrow}$ a is hermitian / ермитов element) then $\rho_{\mathcal{A}}(a) = \|a\|$.

<u>Proof.</u> $\|a^2\| = \|a^* a\| = \|a\|^2$, $(a^2)^* = (a^*)^2 = a^2 \Rightarrow \|a^4\| = \|a\|^4$.
$\cdots \Rightarrow \|a^{2^n}\| = \|a\|^{2^n}$. $\Rightarrow \rho_{\mathcal{A}}(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \|a\|$.

(b) $\|a\| = \rho_{\mathcal{A}}(a^* a)^{1/2}$ for all $a \in \mathcal{A}$.

<u>Since</u> $(a^* a)^* = a^* a$ and $\|a^* a\| = \|a\|^2$.

<u>Remark</u> (b) relates an algebraic quntity $\rho_{\mathcal{A}}(a^* a)^{1/2}$ to a topological quantity $\|\cdot\|$. We will see later, in Corollary 2.11(h) that this implies that the C^*-norm, when it exists, is algebraically determined.

(c) If $a^* a = 1$ ($\overset{def}{\Longleftrightarrow}$ a is isometry / изометрия) or
$a^* a = 1 = a a^*$ ($\overset{def}{\Longleftrightarrow}$ a is unitary / унитарен element)
then $\rho_{\mathcal{A}}(a) = \|a\| = 1$.

<u>Proof.</u> $(a^*)^n a^n = 1$. $\Rightarrow 1 = \|(a^*)^n a^n\| = \|a^n\|^2$. $\Rightarrow \|a^n\| = 1$.
$\Rightarrow \|a\| = 1$ & $\rho_{\mathcal{A}}(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = 1$.

(d) If a is unitary then $\sigma_{\mathcal{A}}(a) \subseteq \{ z \in \mathbb{C} \mid |z| = 1 \}$.

Proof. By (c) : $\sigma_{\mathcal{A}}(a) \subseteq \{ z \in \mathbb{C} \mid |z| \leq 1 \}$.
Since $a^{-1} = a^*$ and $(a - z.1)^{-1} = -z^{-1} a^{-1} (a^{-1} - z^{-1}.1)^{-1}$ then $z \in \sigma_{\mathcal{A}}(a) \Leftrightarrow z^{-1} \in \sigma_{\mathcal{A}}(a)$. $\Rightarrow |z| = 1$ for $z \in \sigma_{\mathcal{A}}(a)$.

(e) If $a = a^*$ (a is hermitian) then $\sigma_{\mathcal{A}}(a) \subseteq [-\|a\|, \|a\|]$.

Proof. By (a): $\sigma_{\mathcal{A}}(a) \subseteq \{ z \in \mathbb{C} \mid |z| \leq \rho(a) = \|a\| \}$.
It remains to prove that $\sigma_{\mathcal{A}}(a) \subseteq \mathbb{R}$, i.e. that if $z \in \mathbb{C} \setminus \mathbb{R}$ then $\exists (a - z.1)^{-1}$. We shall use the following identity

$$\frac{x-\lambda}{x+\lambda} - \frac{y-\lambda}{y+\lambda} = \frac{1}{x+\lambda} \frac{1}{y+\lambda} 2\lambda (x-y)$$

Let $\lambda > \|a\|$ then : replacing $x \mapsto a$, $y \mapsto z$, $\lambda \mapsto i\lambda$

$$\Rightarrow \quad \frac{a - i\lambda 1}{a + i\lambda.1} - \frac{z - i\lambda}{z + i\lambda} \cdot 1 = \frac{1}{a + i\lambda.1} \frac{1}{z + i\lambda} 2i\lambda (a - z.1)$$

and $a - z.1 = (2i\lambda)^{-1} (a + i\lambda.1)(z + i\lambda) \left(\frac{a - i\lambda 1}{a + i\lambda.1} - \frac{z - i\lambda}{z + i\lambda} \cdot 1 \right)$

where $\frac{1}{a + i\lambda.1} := (a + i\lambda.1)^{-1} = R_a(z)$, which exists and is invertible since $|i\lambda| > \|a\|$ and then $-i\lambda \notin \sigma_{\mathcal{A}}(a)$ by (a).

Thus, $z \in \sigma_{\mathcal{A}}(a) \Leftrightarrow \frac{z - i\lambda}{z + i\lambda} \in \sigma_{\mathcal{A}}\left(\frac{a - i\lambda 1}{a + i\lambda.1} \right)$.

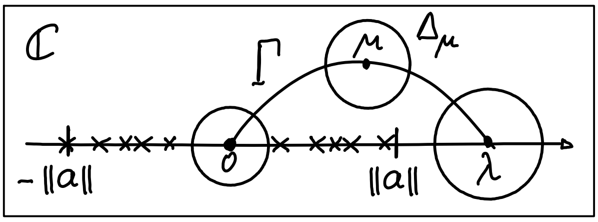
But $\frac{a - i\lambda 1}{a + i\lambda.1}$ is unitary (check!). By (d):

$\sigma_{\mathcal{A}}\left(\frac{a - i\lambda 1}{a + i\lambda.1} \right) \subseteq \{ w \in \mathbb{C} \mid |w| = 1 \}$. On the other hand:

$z \in \mathbb{R} \Leftrightarrow \left| \frac{z - i\lambda}{z + i\lambda} \right| = 1$ (prove!). (For $\lambda = 1$ this is called Cayley/Keňau transformation.) Hence, $z \notin \sigma_{\mathcal{A}}(a)$.

(f) If a is invertible, i.e. $\exists a^{-1}$ then a^{-1} is contained in the smalest closed subalgebra $\mathcal{U}$ of $\mathcal{A}$ which contains $1, a, a^*$ (this is called the closed subalgebra generated by $1, a, a^*$).

<u>Proof.</u> We consider first the case $a = a^*$ (a is hermitian). Then $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a) \supseteq \mathbb{C} \setminus [-\|a\|, \|a\|]$ and $\mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ is a connected open set (why?). $0 \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ by assumptions.



Take $\lambda \in \mathbb{C}$ with $|\lambda| > \|a\|$.
$\Rightarrow \lambda \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ and even more

$R_a(\lambda) \equiv (a - \lambda.1)^{-1} = -\lambda^{-1} \sum_{n=0}^{\infty} a^n \lambda^{-n}$ – absolutely convergent

$\Rightarrow R_a(\lambda) \in \mathcal{U}$. Then take a continuous compact path $\Gamma \subseteq \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$ which connects 0 and λ. As we know from the proof of Theorem 2.5, for every $\mu \in \Gamma$ the series

$$R_a(z) = \left(1 - (z-\mu) R_a(\mu)\right)^{-1} R_a(\mu) = \sum_{n=0}^{\infty} R_a(\mu)^{n+1} (z-\mu)^n$$

is absolutely convergent in some disc $\Delta_\mu \ni z$. So, $R_a(z)$ is contained for $z \in \Delta_\mu$ in every closed subalgebra that contains $R_a(\mu)$. Since Γ is compact we can cover it by finite $\Delta_{\mu_0}, \dots, \Delta_{\mu_n}$ so that $\mu_0 = \lambda$, $\mu_n = 0$ and $\mu_{k+1} \in \Delta_{\mu_k}$ $(k = 0, \dots, n-1)$ (why?). $\Rightarrow R_a(0) = a^{-1} \in \mathcal{U}$.

Now if $a \in \mathcal{A}$ is arbitrary we apply the previous case to a^*a whose inverce is $a^{-1}(a^{-1})^*$ and since $a^{-1} = (a^*a)^{-1} a^*$ we get that $a^{-1} \in \mathcal{U}$.

(g) If $\mathcal{C}$ is a unital C^*-subalgebra of $\mathcal{A}$, i.e. a subalgebra, which is $*$-invariant and closed with respect to the norm, and if $a \in \mathcal{C}$ then for all $z \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(a)$: $R_a(z) \in \mathcal{C}$. In other words,

$$\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{C}}(a)$$

- the spectrum is independent of the algebra.

<u>Proof.</u> We have to prove that if $a - z.1$ is invertible in $\mathcal{A}$ and $a \in \mathcal{C}$ then $(a - z.1)^{-1} \in \mathcal{C}$: this follows from (f).

(f) Let $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras and $\varphi : \mathcal{A} \to \mathcal{B}$ is a morphism of unital $*$-algebras (i.e. only algebraic). Then φ is bounded (i.e., continuous), and $\|\varphi\| = 1$.

Later we shall prove that under the above conditions if φ is a monomorphism (мономорфизм $\overset{def}{\Longleftrightarrow}$ $\operatorname{Ker} \varphi = 0$, i.e. φ is injection) of unital algebras then φ is isometry (изометрия), i.e., $\|\varphi(a)\| = \|a\|$ for all $a \in \mathcal{A}$.

<u>Remark</u> Note we do not require that φ is continuous (i.e., bounded)! φ is just an algebraic morphism preserving the unit ($\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$) and the involution ($\varphi(a^*) = \varphi(a)^*$).

<u>Proof.</u> The inclusion $\sigma_{\mathcal{A}}(a) \supseteq \sigma_{\mathcal{B}}(\varphi(a))$ is obvious since

$$z \in \mathbb{C} \setminus \sigma_{\mathcal{A}}(a) \Rightarrow \exists (a - z.1)^{-1} \in \mathcal{A} \underset{(\text{i.e. } \subseteq)}{\Rightarrow} \exists (\varphi(a) - z.1)^{-1} = \varphi((a - z.1)^{-1}) \in \mathcal{B}$$

$$\Rightarrow z \in \mathbb{C} \setminus \sigma_{\mathcal{B}}(\varphi(a)).$$

Hence for $\forall a \in \mathcal{A}$: $\rho_{\mathcal{A}}(a) = \sup\{|z| \mid z \in \sigma_{\mathcal{A}}(a)\} \geq$
$\geq \sup\{|z| \mid z \in \sigma_{\mathcal{B}}(\varphi(a))\} = \rho_{\mathcal{B}}(\varphi(a))$ and then:

$\|a\| = \rho_{\mathcal{A}}(a^* a)^{1/2} \geq \rho_{\mathcal{B}}(\varphi(a)^* \varphi(a))^{1/2} = \|\varphi(a)\|$. Thus, φ is continuous (bounded) and even $\|\varphi\| \leq 1$, and since $\varphi(1) = 1$ then $\|\varphi\| = 1$. $\square$